result along these lines.

Theorem 14.1. Let $A=(A_1,\ldots,A_n)$ be an n-tuple of operators on a Hilbert space $\mathcal{H}$ such that for all x in $\mathbb{R}^n$,

$$x\cdot A = x_1A_1 + \ldots + x_nA_n$$

is essentially self-adjoint. Then either the $A_1,\ldots,A_n$ commute or there is a ψ in $\mathcal{H}$ with $\|\psi\|=1$ such that there do not exist random variables $\alpha_1,\ldots,\alpha_n$ on a probability space with the property that for all x in $\mathbb{R}^n$ and λ in $\mathbb{R}$,

$$\Pr\{x\cdot\alpha \geq \lambda\} = (\psi, E_\lambda(x\cdot A)\psi),$$

where $x\cdot\alpha=x_1\alpha_1+\ldots+x_n\alpha_n$ and the $E_\lambda(x\cdot A)$ are the spectral projections of the closure of $x\cdot A$.

In other words, n observables may be regarded as random variables, in all states, if and only if they commute.

Proof. We shall not distinguish notationally between $x\cdot A$ and its closure.

Suppose that for each unit vector ψ in $\mathcal{H}$ there is such an n-tuple α of random variables, and let μ_ψ be the probability distribution of α on $\mathbb{R}^n$. That is, for each Borel set B in $\mathbb{R}^n$, $\mu_\psi(B)=\Pr\{\alpha\in B\}$. If we integrate first over the hyperplanes orthogonal to x, we find that

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi}d\mu_\psi(\xi) = \int_{-\infty}^{\infty} e^{i\lambda}d\Pr\{x\cdot\alpha\geq\lambda\}$$

$$= \int_{-\infty}^{\infty} e^{i\lambda}(\psi, dE_\lambda(x\cdot A)\psi) = (\psi, e^{ix\cdot A}\psi).$$

Thus the measure μ_ψ is the Fourier transform of $(\psi, e^{ix\cdot A}\psi)$. By the

polarization identity, if φ and ψ are in $\mathcal{H}$ there is a complex measure $\mu_{\varphi\psi}$ such that $\mu_{\varphi\psi}$ is the Fourier transform of $(\varphi, e^{ix\cdot A}\psi)$ and $\mu_{\psi\psi}=\mu_\psi$. For any Borel set B in $\mathbb{R}^n$ there is a unique operator $\mu(B)$ such that $(\varphi,\mu(B)\psi)=\mu_{\varphi\psi}(B)$, since $\mu_{\varphi\psi}$ depends linearly on ψ and antilinearly on φ. Thus we have

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi}(\varphi, d\mu(\xi)\psi) = (\varphi, e^{ix\cdot A}\psi).$$

The operator $\mu(B)$ is positive since μ_ψ is a positive measure. Consequently, if we have a finite set of elements ψ_j of $\mathcal{H}$ and corresponding points x_j of $\mathbb{R}^n$, then

$$\sum_{j,k} (\psi_k, e^{i(x_j-x_k)\cdot A}\psi_j) =$$

$$\sum_{j,k} \int_{\mathbb{R}^n} e^{i(x_j-x_k)\cdot\xi}(\psi_k, d\mu(\xi)\psi_j) =$$

$$\int_{\mathbb{R}^n} (\psi(\xi), d\mu(\xi)\psi(\xi)) \geq 0,$$

where

$$\psi(\xi) = \sum_j e^{ix_j\cdot\xi}\psi_j.$$

Furthermore, $e^{i0\cdot A}=1$ and $e^{i(-x)\cdot A}=(e^{ix\cdot A})^*$. Under these conditions, the theorem on unitary dilations of Nagy [34, Appendix, p.21] implies that there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a unitary representation $x\to U(x)$ of $\mathbb{R}^n$ on $\mathcal{K}$ such that, if E is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, then

$$EU(x)\psi = e^{ix\cdot A}\psi$$

for all x in $\mathbb{R}^n$ and all ψ in $\mathcal{H}$. Since $e^{ix\cdot A}$ is already unitary,

$$\|U(x)\psi\| = \|e^{ix\cdot A}\psi\| = \|\psi\|,$$

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so that $\|EU(x)\psi\|=\|U(x$
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so that $\|EU(x)\psi\| = \|U(x)\psi\|$. Consequently, $EU(x)\psi = U(x)\psi$ and each $U(x)$ maps $\mathcal{H}$ into itself, so that $U(x)\psi = e^{ix \cdot A}\psi$ for all ψ in $\mathcal{H}$. Since $x \to U(x)$ is a unitary representation of the commutative group $\mathbb{R}^n$, the $e^{ix \cdot A}$ all commute, and consequently the A_j commute. QED.

Quantum mechanics forced a major change in the notion of reality. The position and momentum of a particle could no longer be thought of as properties of the particle. They had no real existence before measurement, and the measurement of the one with a given accuracy precluded the measurement of the other with too great an accuracy, in accordance with the uncertainty principle. This point of view was elaborated by Bohr under the slogan of "complementarity," and Heisenberg wrote a book [35] explaining the physical basis of the new theory.

At the Solvay Congress in 1927, Einstein was very quiet, but when pressed objected that ψ could not be the complete description of the state. For example, the wave function in Fig. 4 would have axial symmetry, but the place of arrival of an individual particle on the hemispherical screen does not have this symmetry. The answer of quantum mechanics is that the symmetrical wave function ψ describes the state of the system before a measurement is made, but the act of measurement changes ψ.

To understand the rôle of probability in quantum mechanics it is necessary to discuss measurement. The quantum theory of measurement was created by von Neumann [33]. A very readable summary is the book by London and Bauer [36]. See also two papers of Wigner [37], [38], which we follow now.